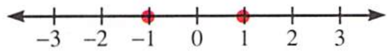
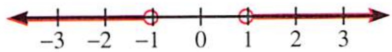
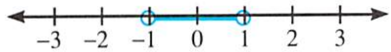
Working with Absolute Value

2-4 Absolute Value in Open Sentences

Objective To solve open sentences involving absolute value.

If you think of the absolute value of a real number x (see page 3) as the distance between the graph of x and the origin on a number line, you can see why the sentences below are equivalent.

Sentence	Equivalent Sentence	Graph
$\lvert x \rvert = 1$ Distance between x and 0 equals 1.	$x = -1$ or $x = 1$	
$\lvert x \rvert > 1$ Distance between x and 0 is greater than 1.	$x < -1$ or $x > 1$	
$\lvert x \rvert < 1$ Distance between x and 0 is less than 1.	$-1 < x < 1$	

You can often solve an open sentence involving absolute value by first writing an equivalent disjunction or conjunction.

Example 1 Solve $|3x - 2| = 8$.

Solution $|3x - 2| = 8$ is equivalent to this disjunction:

$$\begin{aligned} 3x - 2 &= -8 \quad \text{or} \quad & 3x - 2 &= 8 \\ 3x &= -6 \quad \downarrow & 3x &= 10 \\ x &= -2 \quad \text{or} & x &= \frac{10}{3} \end{aligned}$$

$\therefore$ the solution set is $\left\{-2, \frac{10}{3}\right\}$. ***Answer***

Example 2 Solve $|3 - 2t| < 5$.

Solution $|3 - 2t| < 5$ is equivalent to this conjunction:

$$\begin{aligned} -5 < 3 - 2t &< 5 \\ -8 < -2t &< 2 \\ 4 > t &> -1 \end{aligned}$$

$\therefore$ the solution set is $\{t\colon -1 < t < 4\}$. ***Answer***

Example 3 Solve $|2z - 1| + 3 \geq 8$ and graph its solution set.

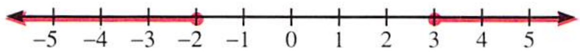
Solution First transform the inequality to an equivalent inequality in which the expression involving absolute value is alone on one side.

$$|2z - 1| + 3 \geq 8$$
$$|2z - 1| + 3 - 3 \geq 8 - 3$$
$$|2z - 1| \geq 5$$

The last inequality is equivalent to this disjunction:

$$2z - 1 \leq -5 \quad \text{or} \quad 2z - 1 \geq 5$$
$$2z \leq -4 \quad \downarrow \quad 2z \geq 6$$
$$z \leq -2 \quad \text{or} \quad z \geq 3$$

$\therefore$ the solution set is $\{z: z \leq -2 \text{ or } z \geq 3\}$. ***Answer***



You can help guard against errors by testing one value from each region of the graph. Substitute values in the original inequality $|2z - 1| + 3 \geq 8$.

Try $z = -4$:	$|2(-4) - 1| + 3 = |-9| + 3 = 12 \geq 8$	True	✓
Try $z = 0$:	$|2 \cdot 0 - 1| + 3 = |-1| + 3 = 4 \geq 8$	False	✓
Try $z = 4$:	$|2 \cdot 4 - 1| + 3 = |7| + 3 = 10 \geq 8$	True	✓

You can tell at a glance that an inequality such as $|x - 3| \geq -2$ is true for all real numbers x, because the absolute value of every real number is nonnegative. On the other hand, an inequality such as $|t + 5| < -1$ has $\emptyset$ as its solution set (why?).

Oral Exercises

Express each open sentence as an equivalent conjunction or disjunction without absolute value.

Sample 1 $|3t - 1| > 2$ **Solution** $3t - 1 < -2$ or $3t - 1 > 2$

1. $|x| \leq 3$ **2.** $|t| = 2$ **3.** $|z| > 0$ **4.** $|y - 3| \leq 2$

5. $|s + 3| = 3$ **6.** $|2x - 3| \geq 1$ **7.** $|3t - 1| \leq 2$ **8.** $|5 - 2z| < 3$

Express each conjunction or disjunction as an equivalent open sentence involving absolute value.

Sample 2 $-1 \leq x - 2 \leq 1$ **Solution** $|x - 2| \leq 1$

9. $u = -3$ or $u = 3$ **10.** $t \geq -3$ and $t \leq 3$ **11.** $3 > 4(x - 1) > -3$

Written Exercises

A **1–8.** Graph the solution set of each open sentence in Oral Exercises 1–8.

Solve and graph the solution set.

9. $|2t + 5| < 3$

10. $|3x + 2| > 4$

11. $|2u - 5| = 0$

12. $8 = |5y + 2|$

13. $\left|1 - \frac{x}{3}\right| \geq \frac{2}{3}$

14. $\left|1 - \frac{p}{2}\right| \leq 2$

15. $0 \leq |4u - 7|$

16. $|3r - 12| > 0$

17. $\left|\frac{t-2}{4}\right| \leq \frac{1}{2}$

18. $1 > |2 - 0.8n|$

Solve.

B **19.** $|x + 5| - 3 = 1$

20. $|2t - 3| + 2 = 5$

21. $|2u - 1| + 3 \leq 6$

22. $4 - |3k + 1| < 2$

23. $7 - 3|4d - 7| \geq 4$

24. $6 + 5|2r - 3| \geq 4$

25. $4 + 2\left|\frac{3t-5}{2}\right| > 5$

26. $2\left|\frac{2t-5}{3}\right| - 3 \geq 5$

27. $7 + 5|c| \leq 1 - 3|c|$

28. $\frac{1}{2}|d| + 5 \geq 2|d| - 13$

Graph the solution set of each open sentence.

C **29.** $2 < |w| < 4$

30. $1 \leq |s - 2| \leq 3$

31. $1 \leq |2x + 1| < 3$

32. $0 < |2 - r| \leq 2$

Solve.

33. $|2x| \leq |x - 3|$

34. $|t| > |2t - 6|$

Computer Exercises

For students with some programming experience.

1. Write a program to list all integers x in the interval $-50 \leq x \leq 50$ that are solutions of an open sentence of the form $a < |cx + d| < b$. The values of a, b, c, and d are to be entered by the user. If no integers in the given interval satisfy the inequality, have the output state this. You will need to use the BASIC function ABS in your program.
2. Use the program in Exercise 1 to find the integer solutions of each open sentence.

   a. $17 < |3x - 25| < 35$ **b.** $1 < |18x + 120| < 100$